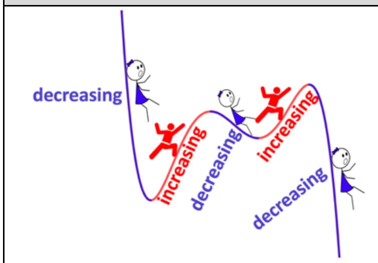
First Derivative $\frac{dy}{dx}$ (talks about slope)

Increasing and decreasing



Increasing:
This is where the slope is positive. Imagine someone climbing up a hill.

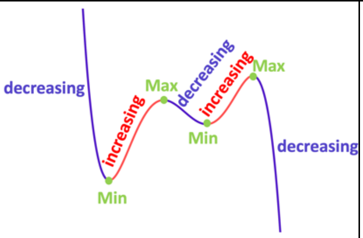
solve $\frac{dy}{dx} > 0$

Decreasing:
This is where the slope is negative. Imagine someone sliding down a hill.

solve $\frac{dy}{dx} < 0$

Important: Remember that when we solve an inequality that is a quadratic or higher we must use the sign change test or graph! We cannot just guess the signs!

Stationary/Turning Points (Max or Min)



Max/Min:
This is where increasing changes to decreasing or vice versa. The slope is neither positive, nor negative, it is zero!

solve $\frac{dy}{dx} = 0$

To verify whether a max or min:

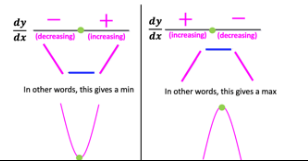
Way 1: Plug into $\frac{d^2y}{dx^2}$

Plug the x value found into $\frac{d^2y}{dx^2}$ and if

$$\frac{d^2y}{dx^2} > 0 \Longrightarrow \text{min}$$
$$\frac{d^2y}{dx^2} < 0 \Longrightarrow \text{max}$$

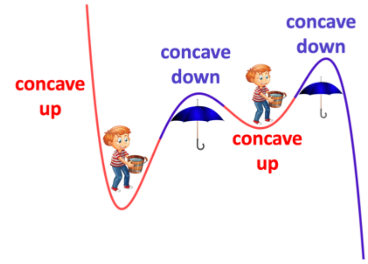
Way 2: Sign change number line test for $\frac{dy}{dx}$

We plug in an x value just below and just above the x value found into $\frac{dy}{dx}$. If $\frac{dy}{dx}$ changes sign from negative $(-)$ to $(+)$ then min and if $\frac{dy}{dx}$ changes from positive $(+)$ to negative $(-)$ then max.



Second Derivative $\frac{d^2y}{dx^2}$ (talks about concavity)

Concave Up (convex) / Concave Down (aka concave)



Concave Up/Convex:
Imagine a bowl or the inside of a bucket. Concave up means the rainwater would be held by the bucket and hence held by the curve.

solve $\frac{d^2y}{dx^2} > 0$

Concave Down/Concave:
Imagine an upside-down bowl or the inside of umbrella. Concave down means the rainwater would roll off and hence would roll off the curve.

solve $\frac{d^2y}{dx^2} < 0$

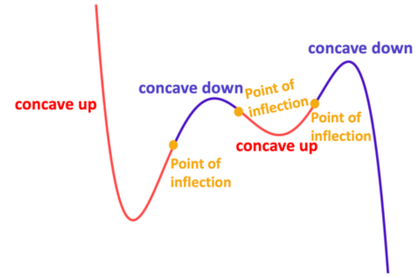
Important: Remember that when we solve an inequality that is a quadratic or higher we should use the sign change test or graph! We cannot just guess the signs!

Note: It should now make sense why we have the criteria

$\frac{d^2y}{dx^2} > 0 \Longrightarrow$ min (holds water)

$\frac{d^2y}{dx^2} < 0 \Longrightarrow$ max (spills water)

Points of Inflection



Points of inflection:
These are where concavity changes from concave up to down or vice versa.
The concavity is neither positive, nor negative, it is zero!

solve $\frac{d^2y}{dx^2} = 0$

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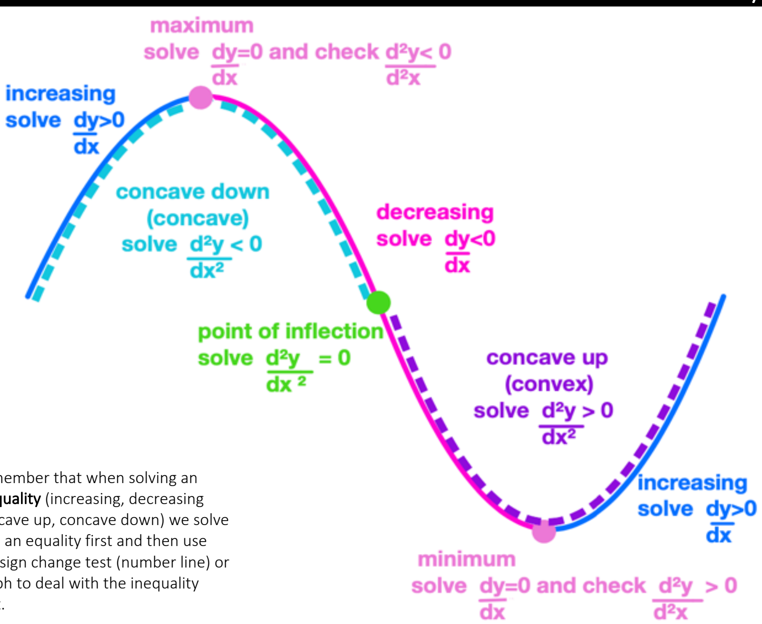
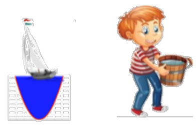
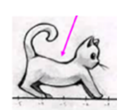
Summary

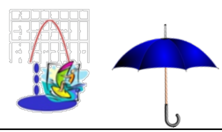
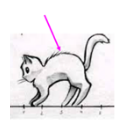


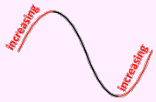
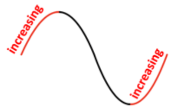
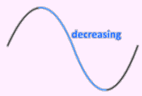
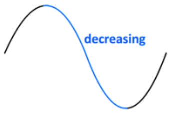
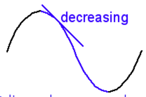
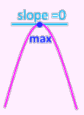
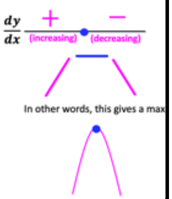
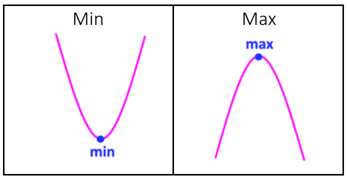
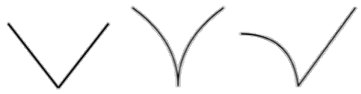
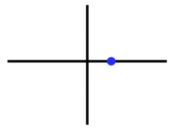
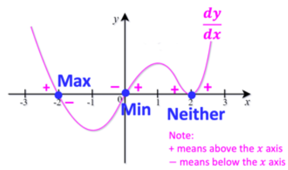
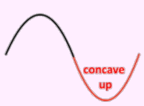
Remember that when solving an **inequality** (increasing, decreasing concave up, concave down) we solve it as an equality first and then use the sign change test (number line) or graph to deal with the inequality part.

Concave Up Looks Like:

holds water	happy cat	yoga - cow pose
		

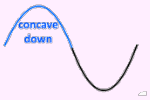
Concave Down Looks Like:

spills water	scared cat	yoga - cat pose
		

Terminology	How to find	y graph	$\frac{dy}{dx}$ graph (1st derivative)	$\frac{d^2y}{dx^2}$ graph (2nd derivative)
Increasing  Slope is positive (going upwards when you look at the curve from left to right). Imagine climbing up a hill.	Solve $\frac{dy}{dx} > 0$ This should make sense since the first derivative gives the slope of a function and we know it is positive when increasing. To solve the inequality, remember to solve as in equality first, plot the values found on a number line and then use the sign change test and look for $+$ region.	graph is going upwards (when looked at from left to right)  To help recognize:  tangent lines drawn are increasing	graph is always ABOVE x axis  graph will be in the turquoise region	We can't tell from this graph
Decreasing  Slope is negative (going downwards when you look at the curve from left to right). Imagine sliding down a hill.	Solve $\frac{dy}{dx} < 0$ This should make sense since the first derivative gives the slope of a function and we know it is negative when decreasing. To solve the inequality, remember to solve as in equality first, plot the values found on a number line and then use the sign change test and look for $-$ region.	graph is going downwards (when looked at from left to right)  To help recognize  tangent lines drawn are decreasing	graph is always BELOW x axis  graph will be in the turquoise region	We can't tell from this graph
Stationary Points/ Turning Points/ Max/Min Stationary and turning points are either maximums or minimums. They occur where the graph changes from increasing to decreasing or vice versa.  or  They occur when the slope is zero (since a horizontal line has zero slope)	Solve $\frac{dy}{dx} = 0$ This should make sense since the first derivative gives the slope of a function and we know the slope is equal to zero when we have a stationary/turning point. To classify whether the stationary/turning points are max or min: **Way 1:** use sign of $\frac{d^2y}{dx^2}$ Plug the x value found into $\frac{d^2y}{dx^2}$. If $\frac{d^2y}{dx^2} > 0 \Longrightarrow$ min $\frac{d^2y}{dx^2} < 0 \Longrightarrow$ max **Way 2:** $\frac{dy}{dx}$ sign change test Plug values either side of the value of x found into $\frac{dy}{dx}$. If $\frac{dy}{dx}$ changes from $-$ to $+ \Longrightarrow$ min $+$ to $- \Longrightarrow$ max  	 Note: when $\frac{dy}{dx}$ is undefined max/min will look like sharp turns (corners/nodes and cusps)  How would we find these points? $\frac{dy}{dx}$ is undefined when the derivative is a fraction and the denominator is equal to zero.	zeros (points on the x axis)  In other words  Min: negative to positive slope Max: positive to negative slope Note: $\frac{dy}{dx} = 0$ doesn't guarantee a max/min (i.e. we can have neither like on the graph above). There must be a sign change of $\frac{dy}{dx}$ (this means + to − or − to +) in order to have a max or min.	We can't tell from this graph
Concave Up (aka convex)  If water was poured on the curve, the curve would hold the water	Solve $\frac{d^2y}{dx^2} > 0$ This should make sense to use the second derivative now, since the first derivative talks about the slope and the second derivative talks about concavity which is positive here. To solve the inequality, remember to solve as in equality first, plot the values found on a number line and then use the sign change test and look for $+$ region.	graph looks like the following: holds water  happy cat  yoga - cow pose  Another way to help recognize:  tangent lines drawn always lie below Note: It should now make sense why $\frac{d^2y}{dx^2} > 0 \Longrightarrow$ min	graph is going upwards 	graph is always ABOVE x axis  graph will be in the turquoise region

Concave Down (aka concave)	Solve $\frac{d^2y}{dx^2} < 0$	graph looks like the following	graph is going downwards	graph is always BELOW x axis

Concave Down (aka concave)



If water was poured on the curve, the curve would spill the water

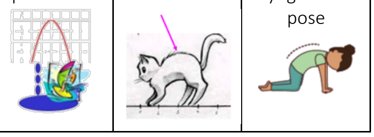
Solve $\frac{d^2y}{dx^2} < 0$

This should make sense since the second derivative talks about concavity (as mentioned in the row above) and our concavity is negative here.

To solve the inequality, remember to solve as in equality first, plot the values found on a number line and then use the sign change test and look for $-$ region.

graph looks like the following

spills water	scared cat	yoga - cat pose

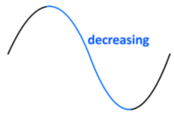


Another way to help recognize:

tangent lines drawn always lie above

Note: It should now make sense why $\frac{d^2y}{dx^2} < 0 \Rightarrow$ max

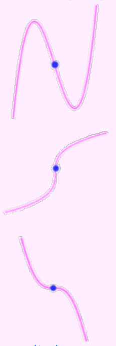
graph is going downwards



graph is always BELOW x axis

graph will be in the turquoise region

Points of Inflection (POI)



concavity is zero (the point where the concavity changes i.e. from concave up to down or vice versa)

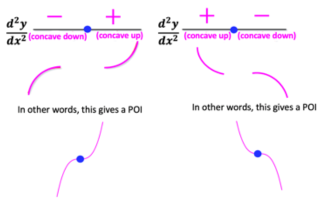
Solve $\frac{d^2y}{dx^2} = 0$

This should make sense since points of inflection are just where concavity changes to from concave up to down or vice versa. The concavity is neither positive, nor negative and hence it is zero.

To prove whether indeed a point on inflection (we need to do this since $\frac{d^2y}{dx^2} = 0$ doesn't guarantee a point of inflection):

Way 1:

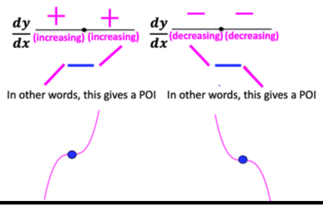
Plug the value found into $\frac{d^2y}{dx^2}$. We need a sign change.



Way 2:

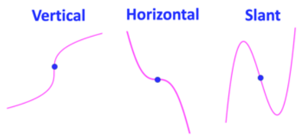
Plug the value found into $\frac{dy}{dx}$.

We need NO sign change.



graph has a change of concavity

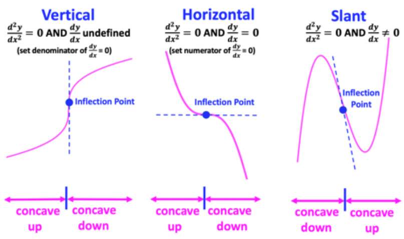
There are 3 types of points on inflection



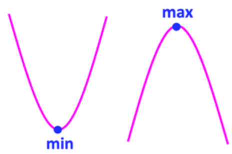
In detail:

$\frac{d^2y}{dx^2} = 0$ will pick up all the following 3 types.

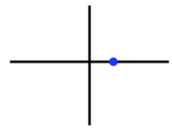
The value of $\frac{dy}{dx}$ allows us differentiates between the 3 types.



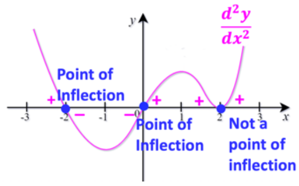
max or mins



zeros (points on the x axis)



In other words

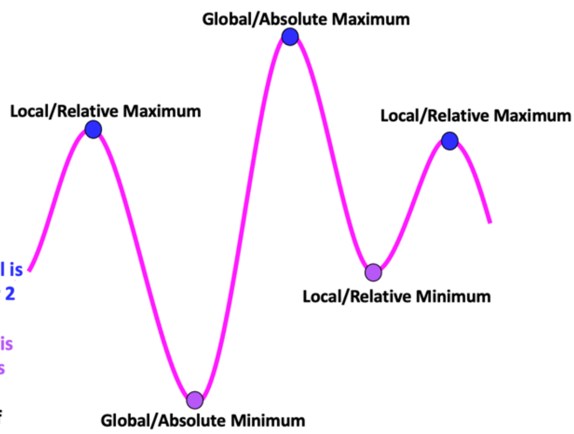


There MUST be a change sign ($+$ to $-$ or $-$ to $+$)

Note: $\frac{d^2y}{dx^2} = 0$ doesn't guarantee a point of inflection, there must be a sign change of $\frac{d^2y}{dx^2}$.

Global/ Absolute vs Relative/Local Min and Max

Global/Absolute vs Relative/Local

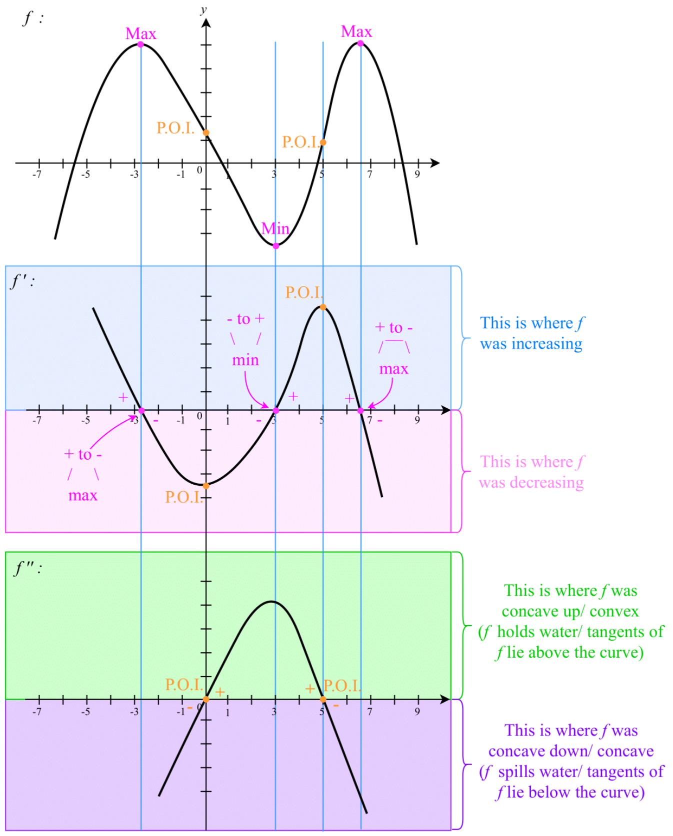


3 max's but highest overall is global/absolute and other 2 are local\relative

2 min's overall but lowest is global/absolute so other is local/relative.

Watch out for endpoints if given them, global/absolute could occur there instead

Drawing f, f′, f″ Graphs:



f to f′ OR f ′ to f ″:

look for whether max/min, inc/dec and POI

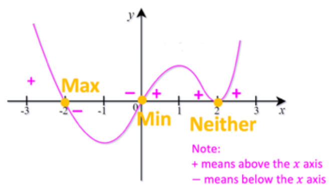
- Max/Min ⟹ zeros
- Inc ⟹ above axis (since greater than zero)
- Dec ⟹ below axis (since less than zero)
- Inflection point ⟹ max or min

Remember that points on inflection on f graph look like:

f′ to f OR f″ to f′ :

look for whether above/below axis, zeros and max/min

- Zeros ⟹ Max/Min
  To know which are min and which are max on original graph we look at how slope changes



- Above axis ⟹ Inc
- Below axis ⟹dec
- Max or Min ⟹ Inflection point

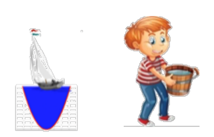
f to f ″:

look for whether inflection points and concavity

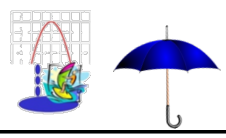
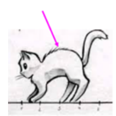
- Inflection point ⟹ zeros
- Concave up ⟹ above axis hence greater than zero
- Concave down ⟹ below axis hence less than zero

Remember what concavity looks like:

Concave up:

holds water	happy cat	yoga - cow pose
		

Concave down:

spills water	scared cat	yoga - cat pose
		

Common Mistakes:

- $\frac{dy}{dx} = 0$ does not mean that we definitely have a min or max!

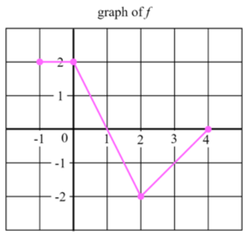
  It could be a point of inflection. Check whether the sign of $\frac{dy}{dx}$ changes either side of the point. if $\frac{dy}{dx}$ doesn't change sign then not a min or max.
- $\frac{d^2y}{dx^2} = 0$ does not mean that we definitely have a point of inflection!

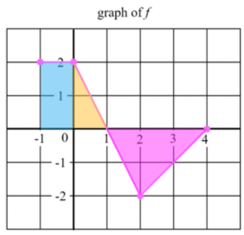
  Check whether the sign of $\frac{d^2y}{dx^2}$ changes either side of the point. If $\frac{d^2y}{dx^2}$ doesn't change sign then not a point of inflection.

Harder Questions Involving Integration:

- $\int y\,dx$ gives us the area under the graph of y. For example,

  The graph shows a piecewise linear function for $-1 \le x \le 4$ is shown in the figure. If the function H is defined by $H(x) = \int_{-1}^{x} f(x)dx$, for $-1 \le x \le 4$. Find $H(4)$.





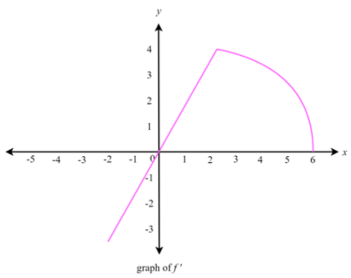
$\int_{-1}^{4} f(x)dx$ = area of rectangle + area of triangle + area of triangle = $1(2) + \frac{1}{2}(1)(2) + -\frac{1}{2}(3)(2) = 2 + 1 - 3 = 0$

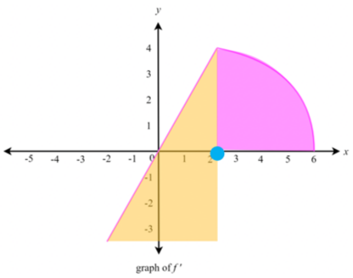
- $\int \frac{dy}{dx} dx$ tells us the area under $\frac{dy}{dx}$, but most importantly tells us how we move up or down on the y graph (what the y value jumps up or down by).

  So, if we're given the $\frac{dy}{dx}$ graph and want to know what the y graph looks like we can find the areas. Being under the x axis on $\frac{dy}{dx}$ graph gives a negative area hence y would decrease and being the over x axis gives a positive area hence y would increase. Normally we are given a starting point. For example,

Example 1:

The graph of $y = f'(x)$, the derivative of a function f, is a line and a quarter circle shown in the diagram. If $f(2) = 3$, find $f(6)$



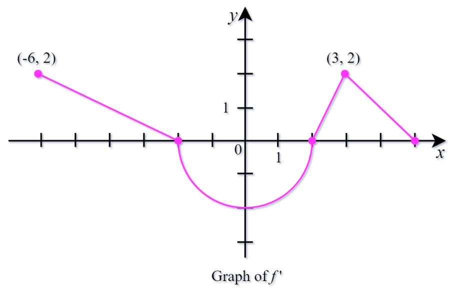


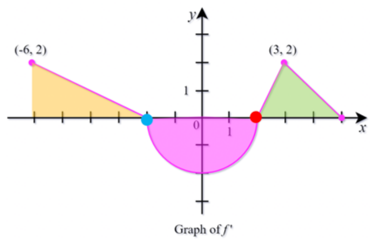
$f(2) = 3$

$f(6)$ = starting value + area of quarter circle = $f(2) + \frac{\pi(4)^2}{4} = 3 + \frac{\pi(4)^2}{4}$

Example 2:

The function f is differentiable on the closed interval $[-6, 5]$ and satisfies f $(-2) = 7$. The graph of $f(x)$, the derivative of f, consists of a semicircle and three line segments, as shown below. Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.





We can see that the value of f would increase from -6 to -2 (since the areas are positive), then decrease from -2 to 2 (since the areas are negative) and then increase again from 2 to 5 (since the areas are positive). The absolute minimum will occur at the endpoint ($x = -6$ or $x = 5$) or at a relative min ($x = 2$)

We are told $f(-2) = 7$

$f(-6)$ = starting value − area of triangle $= 7 - \frac{1}{2}(4)(2) = 3$

(Note: we minus since we are going backwards, so do the opposite to normal)

$f(2)$ = starting value + area of semicircle $= f(-2) - \frac{\pi(2)^2}{2} = 7 - 2\pi$

$f(5)$ = starting value + area of triangle = $f(2) + \frac{1}{2}(3)(2) = 7 - 2\pi + 3$

$= 10 - 2\pi$

The smallest value out of $7,\ 3,\ 7 - 2\pi$ and $10 - 2\pi$ is $7 - 2\pi$

$\therefore$ absolute min value $= 7 - 2\pi$